Int J Theor Phys (2008) 47: 350–356
DOI 10.1007/s10773-007-9456-5

Entropy and Entanglement of a Single-mode Vacuum Field Interacting with a Ξ-type Three-level Atom with Detuning

Jun-Mao Ma · Zhi-Yong Jiao · Ning Li

Received: 22 April 2007 / Accepted: 8 June 2007 / Published online: 11 July 2007

Abstract The evolution of the field entropy and the entanglement between the atom and the field for the system of a single-mode vacuum field interacting with a Ξ-type three-level atom have been studied by using the reduced quantum entropy. The influences of the detuning of the light field and the setting of the initial state of the atom on the field entropy and entanglement of the system under consideration are discussed emphatically. It is showed that the detuning of the light field and the setting of the initial state of the atom play an important role for the evolution of the field entropy and the entanglement between the atom and the field. The general conclusions reached are illustrated by numerical results.

Keywords Single-mode field · Ξ-type three-level atom · Entropy evolution · Entanglement

1 Introduction

The entropy of a radiation field is one of the canonical problems of statistical physics and has attracted much attention in the past. In recent years much attention has been focused on the properties of the entanglement between the field and the atom and in particular the entropy of the system because of the entropy theory about the interaction of the field with the atom presented by Phoenix and Knight (PK) and co-workers [6, 18–20]. The PK entropy theory tells us that the entropy is a very useful and sensitive operational measure of the purity of the quantum state. Because it automatically includes all moments of the density operator. The time behavior of the field (atomic) entropy reflects the time behavior of the entanglement between the atom and the field. The higher the entropy, the greater the entanglement. The information concerning the field is inferred by measurement of atomic properties.

J.M. Ma (✉) · Z.Y. Jiao · N. Li
College of Physics Science and Technology, China University of Petroleum (East China), Qingdao 266555, China
e-mail: jmma2008@163.com

J.M. Ma
e-mail: jmma2008@126.com

On the other hand, Entanglement is one of the key properties distinguishing quantum theory from classical descriptions of the word. Naturally it has become a subject of intensive study in fundamental physics. Recently it has been identified as a central ingredient to facilitate possible practical applications as quantum computation [7], quantum teleportation [4] and quantum dense coding [5]. Therefore, the study for the entropy and entanglement of the system under consideration is very significant.

Some authors have investigated the properties of the field entropy and the entanglement between the field and the atom in the Jaynes-Cummings model, the Tavis-Cummings model and some other models [1, 2, 8–17, 22]. However, less attention has been paid to the properties of the field entropy and the entanglement between the atom and the field for the system of the field interacting with a Ξ-type three-level atom with detuning. In this paper, we will investigate the evolution properties of the field entropy and entanglement of the system of a single-mode vacuum field interacting with a Ξ-type three-level atom and examine the influences of the detuning of the light field and the setting of the initial state of the atom on the field entropy and the entanglement between the field and the atom. Some important results are obtained.

2 The Model and its Solution

The effective Hamiltonian of the model under consideration in this paper in the rotating-wave approximation can be written as

$$H = H_0 + V, \tag{1}$$

where

$$H_0 = \omega a^+ a + \sum_{i=a,b,c} \omega_i c_i^+ c_i \quad (\hbar = 1), \tag{2}$$

$$V = g_1(a c_b^+ c_c + a^+ c_c^+ c_b) + g_2(a c_a^+ c_b + a^+ c_b^+ c_a), \tag{3}$$

where a^+ and a denote creation and annihilation operators of photons respectively, c_i^+ and c_i denote creation and annihilation operators of the atom at ith level respectively, ω and ω_i denote the frequency of the light field and the frequency of eigentransition of the atom at ith level respectively and g_1 and g_2 denote the intensities of interaction between the field and the atom.

For simplicity, we only consider the resonant case ($\omega_a - \omega_c = 2\omega$) and make g_1 and g_2 satisfy the equation $g_2 = \sqrt{2} g_1 = g$.

In the interaction picture, Hamiltonian of the system is

$$V^I = g_1(a c_b^+ c_c e^{i\Delta t} + a^+ c_c^+ c_b e^{-i\Delta t}) + g_2(a c_a^+ c_b e^{i\Delta t} + a^+ c_b^+ c_a e^{-i\Delta t}), \tag{4}$$

where $\Delta = \omega - (\omega_b - \omega_c) = (\omega_a - \omega_b) - \omega$ ($\Delta \ll \omega$) is detuning parameter of the light field.

We suppose that the light field is initially in a single-mode vacuum field $|0\rangle$ and the atom is initially in a coherent superposition state of the excited state $|a\rangle$ and the ground state $|c\rangle$

$$|\psi_A(0)\rangle = \cos(\theta/2)|a\rangle + \sin(\theta/2)|c\rangle, \tag{5}$$

where $0 \le \theta \le \pi$ is the setting parameter of the initial state of the atom. It denotes the atomic distribution.

No interaction between the atom and the light field exists at the initial time $t = 0$, so that the initial state vector of the system is written as

$$|\psi_{AF}(0)\rangle = |\psi_A(0)\rangle \otimes |0\rangle = \cos(\theta/2)|a,0\rangle + \sin(\theta/2)|c,0\rangle. \tag{6}$$

As the time goes, the evolution of the system in the interaction picture is governed by the state vector

$$|\psi_{AF}(t)\rangle = C_4(t)|a,0\rangle + C_3(t)|b,1\rangle + C_2(t)|c,2\rangle + C_1(t)|c,0\rangle. \tag{7}$$

Solving the Schrodinger equation

$$i\frac{\partial}{\partial t}|\psi_{AF}(t)\rangle = V^I|\psi_{AF}(t)\rangle, \tag{8}$$

we can get

$$C_1(t) = \sin(\theta/2), \tag{9}$$

$$C_2(t) = \frac{g^2}{\Omega^2}\cos(\theta/2)e^{-i\Delta t}(\cos\Omega t - 1), \tag{10}$$

$$C_3(t) = \frac{g}{\Omega^2}\cos(\theta/2)[\Delta(\cos\Omega t - 1) - i\Omega\sin\Omega t], \tag{11}$$

$$C_4(t) = \frac{1}{\Omega^2}\cos(\theta/2)e^{i\Delta t}[(\Delta^2 + g^2)\cos\Omega t - i\Delta\Omega\sin\Omega t + g^2], \tag{12}$$

where

$$\Omega = \sqrt{\Delta^2 + 2g^2}. \tag{13}$$

3 The Evolution of the Field Entropy and Entanglement of the System

Since we have assumed that the Ξ-type three-level atom and the single-mode vacuum field are initially in a disentangled pure state, the total entropy of the system is equal to zero. In terms of the triangle inequality of the entropy [3]

$$|S_A(t) - S_F(t)| \le S_{AF}(t) \le |S_A(t) + S_F(t)|, \tag{14}$$

we can see that the reduced entropies of the two subsystems are identical, namely $S_A(t) = S_F(t)$. Hence, the field entropy can be obtained by operating the atomic entropy.

The reduced density matrix of the atom is given by

$$\rho_A(t) = Tr_F|\psi_{AF}(t)\rangle\langle\psi_{AF}(t)| = \begin{pmatrix} C_4^*C_4 & 0 & C_1^*C_4 \\ 0 & C_3^*C_3 & 0 \\ C_4^*C_1 & 0 & C_1^*C_1 + C_2^*C_2 \end{pmatrix} \tag{15}$$

and the field entropy is given by [21]

$$S_F(t) = S_A(t) = -Tr_A[\rho_A(t)\ln\rho_A(t)] = -\sum_{j=1}^{3}\lambda_j\ln\lambda_j, \tag{16}$$

where λ_j $(j = 1, 2, 3)$ is the eigenvalue of reduced density matrix of the atom. They are determined by the matrix elements of the reduced density matrix of the atom

$$\lambda_1 = C_3^* C_3, \tag{17}$$

$$\lambda_2 = \frac{1}{2}[C_4^* C_4 + C_2^* C_2 + C_1^* C_1 + C], \tag{18}$$

$$\lambda_3 = \frac{1}{2}[C_4^* C_4 + C_2^* C_2 + C_1^* C_1 - C], \tag{19}$$

where

$$C = \sqrt{(C_4^* C_4)^2 + (C_1^* C_1 + C_2^* C_2)(C_1^* C_1 + C_2^* C_2 - 2C_4^* C_4) + 4C_1^* C_1 C_4^* C_4}. \tag{20}$$

In (16), the field entropy $S_F(t)$ reflects the degree of the entanglement between the atom and the field. If $S_F(t)$ takes its minimal value zero, the atom and the field are disentangled. If $S_F(t)$ takes its nonzero value, the atom and the field are entangled.

In order to find the evolution properties of the field entropy and the entanglement between the atom and the field for the system under consideration, different values of Δ and θ are adopted to solve (16) numerically and the results are shown in Figs. 1 and 2.

Figure 1 shows the influences of the setting parameter of the initial state of the atom on the evolution properties of the field entropy and entanglement of the system under consid-

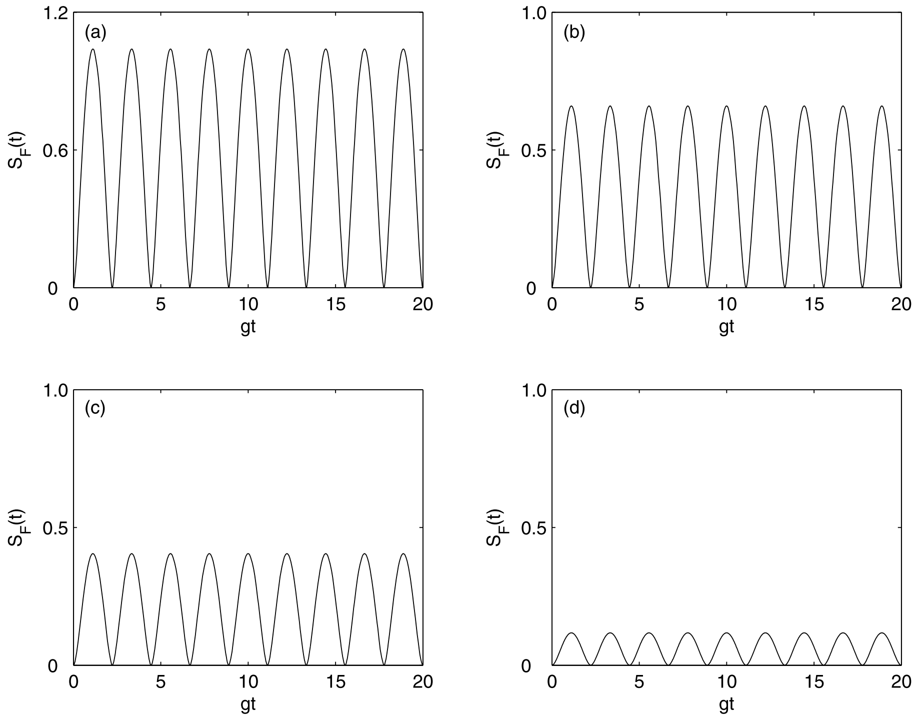

Fig. 1 The evolution properties of the field entropy and entanglement of the system under consideration, for $\Delta = 0$, **a** $\theta = 0$; **b** $\theta = \frac{\pi}{2}$; **c** $\theta = \frac{2\pi}{3}$; **d** $\theta = \frac{6\pi}{7}$

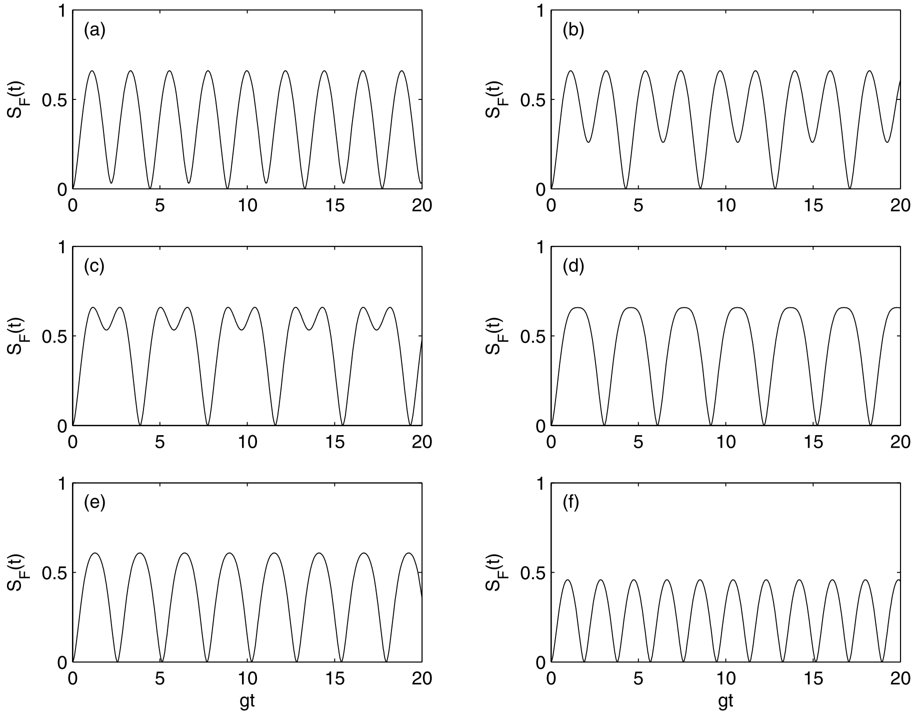

Fig. 2 The evolution properties of the field entropy and entanglement of the system under consideration, for $\theta = \frac{\pi}{2}$, **a** $\Delta = 0.1g$; **b** $\Delta = 0.4g$; **c** $\Delta = 0.8g$; **d** $\Delta = 1.5g$; **e** $\Delta = 2g$; **f** $\Delta = 3g$

eration. From these four pictures, it is observed that the field entropy exhibits periodic oscillation. With the increase of θ, the maximal values of the field entropy gradually decrease. But the evolutional period of the field entropy has not changed. The results correspond with the fact that it happens to the maximal entanglement and disentanglement between the atom and the field periodically and the degree of the maximal entanglement between the atom and the field gradually decreases with the increase of θ. In other words, that the atom is initially in the excited state $|a\rangle$ is advantageous to result in greater entanglement between the atom and the field.

Figure 2 shows the influences of the detuning parameter of the light field on the evolution properties of the field entropy and entanglement of the system under consideration. From these six pictures, it is found that the field entropy exhibits periodic oscillation. When the detuning parameter is very small, there is no change for the maximal values of the field entropy. But there is a slight change for the evolutional period of the field entropy. Meanwhile, there appear some sub-minimal values of the field entropy. These results are found from Figs. 2a–c. When the detuning parameter is enough large, the maximal values of the field entropy gradually decrease with the increase of the detuning parameter. There is a obvious change for the evolutional period of the field entropy. At the same time, sub-minimal values of the field entropy disappear. These results are found from Figs. 2d–f. The above results correspond with the fact that it happens to the maximal entanglement and disentanglement between the atom and the field periodically and the periods of the entanglement and disentanglement between the atom and the field become small with the increase of the detuning parameter. When the detuning parameter is very small, the degree of the maximal entanglement between the atom and the field doesn't change. When the detuning parameter is enough large, the degree of the maximal entanglement between the atom and the field gradually decreases with the increase of the detuning parameter.

4 Conclusions

In this paper, we have studied the evolution properties of the field entropy and the entanglement between the atom and the field for the system of a single-mode vacuum field interacting with a Ξ-type three-level atom by using the reduced quantum entropy and examined the influences of the detuning parameter of the light field and the setting parameter of the initial state of the atom on the field entropy and entanglement of the system under consideration. The results can be concluded as: the detuning parameter of the light field and the setting parameter of the initial state of the atom have an important effect on the evolution properties of the field entropy and the entanglement between the atom and the light. In the whole evolutionary process, the field entropy exhibits periodic oscillation and it happens to the maximal entanglement and disentanglement between the atom and the field periodically. For small values of the detuning parameter, the changes of the detuning parameter hardly affect the maximal values of the field entropy, but affect the evolutional period of the field entropy slightly. The degree of the maximal entanglement between the atom and the field hardly changes. While for large values of the detuning parameter, there are several obvious changes for the maximal entropy and evolutional period of the field entropy. With the increase of the detuning parameter, the maximal values of the field entropy gradually decrease and the evolutional period of the field entropy becomes small. The degree of the maximal entanglement between the atom and the field also starts decreasing. On the other hand, with the setting parameter of the initial state of the atom increasing, there is an obvious decrease for the maximal value of the field entropy and the degree of the maximal entanglement between the atom and the field. This result corresponds with the fact that the atom is initially in the excited state is advantageous to result in greater entanglement between the atom and the field. But these is no change for the evolutional period of the field entropy. Ultimately, we can control the field entropy and the entanglement between the atom and the field by choosing the appropriate detuning parameter and setting parameter of the initial state of the atom. It may be very important for the experimental realization of the preparation of entangled states.

Acknowledgements This work was partly supported by the Science Foundation of China University of Petroleum under Grant No Y061815.

References

1. Abdel-Aty, M., Al-Kader, G.M.A., Obada, A.-S.F.: Chaos Solitons Fractals **12**, 2455 (2001)
2. Abdel-Aty, M., Abdel-Khalek, S., Obada, A.-S.F.: Chaos Solitons Fractals **12**, 2015 (2001)
3. Araki, H., Lieb, E.H.: Commun. Math. Phys. **18**, 160 (1970)
4. Bennett, C.H., Brassard, G., Crepeau, C., Jozsa, R., Peres, A., Wootters, W.K.: Phys. Rev. Lett. **70**, 1895 (1993)
5. Braunstein, S.L., Kimble, H.J.: Phys. Rev. A **61**, 042302 (2000)
6. Buzek, V., Moya-Cessa, H., Knight, P.L., Phoenix, S.J.D.: Phys. Rev. A **45**, 8190 (1992)
7. Cirac, J.I., Zoller, P.: Phys. Rev. Lett. **74**, 4091 (1995)
8. Fang, M.F., Liu, X.: Phys. Lett. A **210**, 11 (1996)

9. Fang, M.F., Zhou, P.: Phys. A **234**, 571 (1996)
10. Fang, M.F., Zhu, S.Y.: Phys. A **369**, 475 (2006)
11. Gao, Y.F., Feng, J., Wang, J.S.: Chin. Phys. **14**, 980 (2005)
12. Huang, C.J., Tang, L.J., Kong, F.Z., Fang, J.Y., Zhou, M.: Phys. A **368**, 25 (2006)
13. Jin, L.J., Fang, M.F.: Chin. Phys. **15**, 2012 (2006)
14. Liao, X.P., Fang, M.F., Zhou, Q.P.: Phys. A **365**, 351 (2006)
15. Liu, T.K., Wang, J.S., Feng, J., Zhan, M.S.: Chin. Phys. **14**, 536 (2005)
16. Liu, T.K.: Chin. Phys. **15**, 542 (2006)
17. Obada, A.-S.F., Mohammed, F.A., Hessian, H.A., Mohammed, A.-B.A.: Int. J. Theor. Phys. **46**, 1027 (2007)
18. Phoenix, S.J.D., Knight, P.L.: Phys. Rev. A **44**, 6023 (1991)
19. Phoenix, S.J.D., Knight, P.L.: Phys. Rev. Lett. **66**, 2833 (1991)
20. Phoenix, S.J.D., Knight, P.L.: Ann. Phys. **186**, 381 (1988)
21. Vedral, V., Plenio, M.B., Rippin, M.A., Knight, P.L.: Phys. Rev. Lett. **78**, 2275 (1997)
22. Zidan, N.A., Abdel-Aty, M., Obada, A.-S.F.: Chaos Solitons Fractals **13**, 1421 (2002)